On the dispersive properties of eigenfunctions

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General setting

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We'll focus on the simplest case where M is a compact manifold without boundary of dimension 2; however, many of the results hold in greater generality.

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Resulting eigenspace, "spherical harmonics of degree k",

$$\mathcal{H}_k = \{\, e_{k,1}, e_{k,2}, \dots, e_{k,d_k} \,\},$$

$$-\Delta_{S^2}\, e_{k,j} = (k^2+k)e_{k,j}$$

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Calculate: $\|Z_k\|_{L^p(S^2)} \approx \lambda_k^{2(1/2-1/p)-1/2}$, $p \geq 6$.

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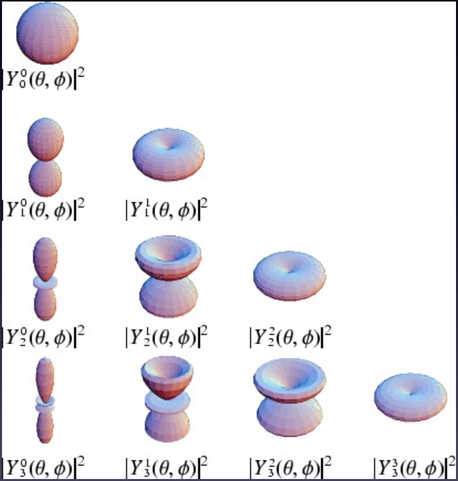
Also, clearly $\|Q_k\|_p \approx \lambda_k^{\frac{1}{2}(\frac{1}{2}-\frac{1}{p})}, \;\; p\ge 2$.

Spherical harmonics



Saturation of L^p norms on S^2

In my thesis with Stein ('85), showed that eigenfunctions on S^2 satisfy

$$\|e_k\|_{L^p(S^2)} \lesssim \lambda_k^{\delta(p)} \|e_k\|_{L^2(S^2)},$$

with

$$\delta(p) = \begin{cases} 2(1/2 - 1/p) - 1/2, & p \geq 6 \\ \frac{1}{2}(\frac{1}{2} - \frac{1}{p}), & 2 \leq p \leq 6. \end{cases}$$

Conclude that

- Zonal functions, Z_k, saturate L^p norms for "large" p (i.e., $p \geq 6$)
- Highest weight spherical harmonics, Q_k, saturate norms for "small" p (i.e., $2 \leq p \leq 6$)

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- Get improved bounds for $p>6$ if at every $x\in M$ there is zero measure of closed loops thru x...

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if and only if

$$\sup_{\gamma \in \Pi} \int_{\mathcal{T}_{\lambda_j^{-\frac{1}{2}}}(\gamma)} |e_{\lambda_j}(x)|^2 \, dV_g = o(1) \qquad (3)$$

Remarks

Shrinking L^2-mass, $\sup_{\gamma\in\Pi}\int_{\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)} |e_\lambda(x)|^2\, dV_g = o(1)$, is antithesis of what happened for the highest weight spherical harmonics, Q_k

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Bourgain '09: $\int_\gamma |e_\lambda|^2\, ds \lesssim \lambda^{\frac{1}{p}} \|e_\lambda\|^2_{L^p(M)},\ p \geq 2$

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Punch line: Improving $L^p(M)$ estimates in (open) range where highest weight spherical harmonics saturate exactly equivalent to improving restriction estimate of BGT:

$$\int_\gamma |e_\lambda|^2 ds \leq C\lambda^{\frac{1}{2}} \|e_\lambda\|_2^2, \quad \gamma \in \Pi. \tag{4}$$

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- C.S. Can improve (4) if γ not part of a periodic geodesic

Improved BGT $\implies o(\lambda^{\delta(p)})$-$L^p(M)$

Enough to consider $p = 4$. Showed that there is a constant C so that for $N = 1, 2, 3, \dots$

$$\int_M |e_\lambda(x)|^4 \, dV_g \le C N^{-1/2} \lambda^{1/2} \|e_\lambda\|^4_{L^2(M)} + C N \lambda^{1/2} \|e_\lambda\|^2_{L^2(M)} \Big[\sup_{\gamma \in \Pi} \int_{\mathcal{T}_{\lambda^{-1/2}}(\gamma)} |e_\lambda(x)|^2 \, dV_g \Big] \quad (5)$$

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- Ours, uses these ideas together with alternate proof of Hörmander (oscillatory integrals) and Gauss' lemma

Improved results for ≤ 0 curv in 2-d

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Bérard in the 70's (implicitly) showed that

$$\|e_\lambda\|_{L^\infty(M)} = O(\lambda^{1/2}/(\log\lambda)^{1/2}) \implies \|e_\lambda\|_{L^p(M)} = o(\lambda^{\delta(p)}), \forall p > 6$$

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Improved results for ≤ 0 curv in 2-d

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Unknown about whether there's o-results for $L^6(M)$ in 2-d, even under assumption of $<$ curvature. Interesting problem.

Improved restriction bounds:

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$$e_\lambda = \frac{1}{2\pi T} \int_0^T \hat{\chi}(t/T) e^{it\lambda} \cos(t\sqrt{-\Delta_g}) e_\lambda \, dt + \chi(T(\lambda + \sqrt{-\Delta_g}))e_\lambda.$$

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Last term is $O(\lambda^{-N})$, and so want, under the ≤ 0 curvature assumption, improvement from **time-averaging**:

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some $\sigma > 0$. A type of "dispersion" for wave equation for non-positive curvature (false for S^2).

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Can work in geodesic coordinates and assume

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Above a nice FIO (loc canonical graph when acting on such f). This and wave-front set analysis allows you, in above time-averaging integral, to cut away all times not near multiples of primitive period, $\ell(\gamma)$, of our $\gamma \in \Pi_{periodic}$, **and more**

Hadamard: If (M,g) has ≤ 0 curvature and $x_0 \in \gamma \subset M$ then $p = \exp_{x_0} : T_{x_0}M \simeq \mathbb{R}^2 \to M$ is a universal cover. Let Γ be the set of deck transformations (homeomorphisms α s.t. $p \circ \alpha = p$).

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If $\tilde g$ is the pullback of g then solutions of wave equations, $u(t,x)$, on (M,g) correspond exactly to periodic ones on universal cover (i.e. $\tilde u(t,\alpha(\tilde x)) = \tilde u(t,\tilde x)$, $\alpha \in \Gamma$). Thus,

$$\big(\cos t\sqrt{-\Delta_g}\big)(x,y) = \sum_{\alpha\in\Gamma} \big(\cos t\sqrt{-\Delta_{\tilde g}}\big)(\tilde x, \alpha(\tilde y)), \text{ if } p(\tilde x) = x,\ p(\tilde y) = y.$$

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I.e., just need to consider $O(T)$ of above terms, as in geodesic polar coordinates, $ds^2 = dr^2 + \rho(r, \theta)d\theta^2$. Can "throw away" rest, by FIO facts from previous slide.

Choose fund domain $D \subset \mathbb{R}^2$ for M. For $x \in M$, $\exists! \, \tilde{x} \in D$ so that $p(\tilde{x}) = x$. Then are reduced to showing

$$\sum_{\substack{\alpha \in \text{Stab}(\tilde{\gamma}_0) \\ d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y})) \le T}} \Big\| \frac{1}{T} \iint \hat{\chi}(\tfrac{t}{T}) e^{it\lambda} \cos t\sqrt{-\Delta_{\tilde{g}}}(\tilde{x}, \alpha(\tilde{y})) f(y) \, dt dV_g(y) \Big\|_{L^2(x \in \gamma)}$$

$$\le CT^{-\sigma} \lambda^{\frac{1}{4}} \|f\|_{L^2(M)}, \ \lambda \text{ large.}$$

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Hadamard Parametrix:

$$\big(\cos(t\sqrt{-\Delta_{\tilde{g}}})\big)(\tilde{x}, \alpha(\tilde{y}))$$

$$= a_0(\tilde{x}, \alpha(\tilde{y}))(2\pi)^{-2} \int_{\mathbb{R}^2} e^{iz\cdot\xi} \cos t|\xi|\, d\xi + Better, \text{ if } |z| = d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y})).$$

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Using $O(|z|^{-\frac{1}{2}})$ decay of Fourier integral, easy to see get above with $\sigma = 1/2$ if principal Hadamard coefficient satisfies

$$a_0(\tilde{x},\alpha(\tilde{y})) = O(1)$$

Last miracle: Vol comparison bounds

If you write dV_g in geodesic polar coordinates about $\tilde{x}$,

$$dV_g(\tilde{z}) = \rho(r,\theta)drd\theta, \ r = d_{\tilde{g}}(\tilde{x},\tilde{z}),$$

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(and, *even better*, $\rho \ge \frac{1}{\kappa}\sinh(\kappa r)$, if curvature $\le -\kappa^2$, with $\kappa > 0$.)

Thank You!


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